Transport of energy and momentum by gravitational waves from a rotating rod: the linear approximation

1968 J. Phys. A: Gen. Phys. 1 97

(http://iopscience.iop.org/0022-3689/1/1/313)

J. PHYS. A (PROC. PHYS. SOC.), 1968, SER. 2, VOL. 1. PRINTED IN GREAT BRITAIN

Transport of energy and momentum by gravitational waves from a rotating rod: the linear approximation

M. A. ROTENBERG
The Negev Institute for Arid Zone Research, Beersheva, Israel
MS. received 28*th June* 1967

Abstract. Gravitational waves emitted from a spinning rod—taken as a rigid axially symmetric distribution of matter of uniform small cross section—are studied within the framework of general relativity, with special interest on transport of linear momentum from the rotating source. It is found that linear momentum is carried by the waves cyclically in such a way that the centre of rotation is a fixed point *outside* the axis of symmetry of the rod.

Two results for the rate of momentum flux, differing in numerical content, are derived from the solution of the linear approximation, one by the use of Synge's argument involving the energy–momentum tensor, and the other by means of the pseudo-tensor. The discrepancy between the results is explained.

1. Introduction

Transport of energy by gravitational waves from a *rotating* source has been fairly extensively investigated during the last few decades since the invention of general relativity (Einstein 1916, 1918, Eddington 1924, p. 248, Clark 1947, Weber 1961, § 7.6). That a spinning rod loses energy at a steady rate as gravitational radiation is a familiar result. It has also been shown that a bounded cohesive rotating source, such as the rod or an ellipsoid, in general gradually loses its angular momentum (Clark 1947). The question that seems not to have been previously studied is whether gravitational waves carry *linear* momentum from a rotating source, thus causing movement of the centre of mass of the source. If linear momentum transport does occur, then it is expected to be of a cyclic nature of a kind that will make the source rotate about a fixed point not coincident with its centre of mass. The main object of this work is to establish the existence of transport of linear momentum of this sort by gravitational waves from a rotating rod, starting with the solution of the linear approximation to the gravitational field equations†

$$R_{ik} = 0 \tag{1.1}$$

for free space, using two methods: (i) Synge's argument which involves the energy–momentum tensor T_{ik}; (ii) the pseudo-tensor $\mathfrak{t}_k{}^i$.

The leading moments of the energy tensor for the spinning rod are evaluated in § 2 and are needed in § 3 to derive the solution, for the rod, of the linear approximation to (1.1). Using the two methods enunciated above, in § 4 alternative values for the 4-momentum flux of gravitational radiation from the spinning rod are calculated, and then the difference between the two values is explained.

2. Calculation of the leading moments of the energy tensor for the rotating rod

In the linear approximation to (1.1) we shall assume that distance, time and mass have their Newtonian meanings.

We choose a (pseudo-) rectangular Cartesian coordinate system Oxyz in which the rod A_1OA_2 of mass m, length a and small uniform cross section $\tilde{s}$ rotates with constant angular velocity ω in the plane Oxy about the origin O. In the linear approximation we shall

† In this paper, unless otherwise stated or implied, a roman index ranges from 1 to 4, as in (1.1), and a Greek one from 1 to 3; the summation convention applies to both forms of index.

assume that the centre of mass of the rod coincides with O.† Let us choose the origin of time t such that the axis of symmetry of the rod coincides with the axis Ox at $t = 0$; thus, at time t, $\widehat{x\mathrm{OA}_2} = \omega t$.

We write $\mathrm{OA}_i = ak_i$ $(i = 1, 2)$ and consider any point Q on the rod, so that $\mathrm{OQ} = a\zeta\,(-k_1 \leqslant \zeta \leqslant k_2)$. Let ρ be the volume density at Q. Then, from elementary dynamics it is readily found that the stress at Q at time t is given by

$$p = ma\omega^2\tilde{s}^{-1}h \tag{2.1}$$

where

$$h \overset{\text{def}}{=} \int_{-k_1} \zeta\sigma(\zeta)\,d\zeta \tag{2.2}$$

$$\sigma \overset{\text{def}}{=} \frac{a\tilde{s}}{m}\rho = \frac{\rho}{\displaystyle\int_{-k_1}^{k_2} \rho(\zeta)\,d\zeta} \tag{2.3}$$

are dimensionless quantities independent of m and a. Knowing this stress at Q and the velocity of Q at time t we can proceed to calculate for the rod the quantities

$$I_{ik/\sigma\rho\tau\ldots}(t) \overset{\text{def}}{=} \int_V \tilde{x}_\sigma \tilde{x}_\rho \tilde{x}_\tau \ldots T_{ik}(\tilde{x}_\alpha, t)\,d\tilde{x}_1\,d\tilde{x}_2\,d\tilde{x}_3 \tag{2.4}$$

V being any fixed space volume enclosing the rod. These quantities are the moments, at time t about the coordinate planes, of the energy tensor T_{ik} corresponding to any material distribution. We use the formula (Eddington 1924, § 53, Bergmann 1942, p. 127)

$$T^{ik} = t^{ik} + \rho u^i u^k, \qquad t^{\alpha\beta} = p^{\alpha\beta}, \qquad t^{\alpha 4} = t^{44} = 0 \tag{2.5}$$

which, in the linear approximation and in Galilean coordinates x_i, expresses T^{ik} for any material distribution in terms of the stress tensor $p^{\alpha\beta}$, volume density ρ, and the 4-velocity u^i $(= u^\alpha, 1)$ of that particle of matter which passes through the field point x_α at time x_4.

Calculating (for the rotating rod) the *dimensionless* moments $h_{ik/\sigma\rho\tau\ldots}$ of T_{ik}, defined by

$$\left.\begin{aligned} h_{\alpha\beta/\sigma_1\sigma_2\ldots\sigma_s} &= \frac{I_{\alpha\beta/\sigma_1\sigma_2\ldots\sigma_s}}{ma^{s+2}} \\ h_{\alpha 4/\sigma_1\sigma_2\ldots\sigma_s} &= \frac{I_{\alpha 4/\sigma_1\sigma_2\ldots\sigma_s}}{ma^{s+1}} \\ h_{44/\sigma_1\sigma_2\ldots\sigma_s} &= \frac{I_{44/\sigma_1\sigma_2\ldots\sigma_s}}{ma^{s}} \end{aligned}\right\} \tag{2.6}$$

by the method outlined above, we find for the leading non-zero components the following values:

$$h_{44} = 1 \tag{2.7}$$

$$\left.\begin{array}{llll} h_{11} = -\omega^2\overset{2}{h}(1-2s^2), & h_{12} = -2\omega^2\overset{2}{h}sc, & h_{22} = \omega^2\overset{2}{h}(1-2s^2) & \\ h_{14/1} = \omega\overset{2}{h}sc, & h_{14/2} = \omega\overset{2}{h}s^2, & h_{24/1} = -\omega\overset{2}{h}c^2, & h_{24/2} = -\omega\overset{2}{h}sc \\ h_{44/11} = \overset{2}{h}c^2, & h_{44/12} = \overset{2}{h}sc, & h_{44/22} = \overset{2}{h}s^2 & \end{array}\right\} \tag{2.8}$$

† Admittedly, it is our primary purpose to establish a difference between the centre of rotation O and the centre of mass. Nevertheless, this difference is of an order pertaining to the non-linear approximation to (1.1), although its calculation is based on the solution of the linear approximation. In this approximation it is therefore correct to assume that the centre of mass is the centre of rotation O.

$$\left.\begin{array}{lll}
h_{11/1} = \omega^2 \overset{3}{h}(c-\tfrac{3}{2}c^3), & h_{11/2} = -\omega^2 \overset{3}{h}(\tfrac{1}{2}s-\tfrac{3}{2}s^3), & h_{12/1} = -\tfrac{3}{2}\omega^2 \overset{3}{h} sc^2 \\
h_{12/2} = -\tfrac{3}{2}\omega^2 \overset{3}{h} s^2 c, & h_{22/1} = -\omega^2 \overset{3}{h}(\tfrac{1}{2}c-\tfrac{3}{2}c^3), & h_{22/2} = \omega^2 \overset{3}{h}(s-\tfrac{3}{2}s^3) \\
h_{14/11} = \omega \overset{3}{h} sc^2, & h_{14/12} = \omega \overset{3}{h} s^2 c, & h_{14/22} = \omega \overset{3}{h} s^3 \\
h_{24/11} = -\omega \overset{3}{h} c^3, & h_{24/12} = -\omega \overset{3}{h} sc^2, & h_{24/22} = -\omega \overset{3}{h} s^2 c \\
h_{44/111} = \overset{3}{h} c^3, & h_{44/112} = \overset{3}{h} sc^2, & h_{44/122} = \overset{3}{h} s^2 c \\
h_{44/222} = \overset{3}{h} s^3. & &
\end{array}\right\} \quad (2.9)$$

In these

$$s = \sin \omega t, \qquad c = \cos \omega t; \tag{2.10}$$

$\overset{n}{h}$ $(n = 0, 1, 2, \ldots)$ are dimensionless quantities independent of m and a and given by

$$\overset{n}{h} \overset{\text{def}}{=} \frac{\overset{n}{I}}{ma^n} = \int_{-k_1}^{k_2} \zeta^n \sigma(\zeta)\, d\zeta \tag{2.11}$$

where $\overset{n}{I}$ $(n = 0, 1, 2, \ldots)$ are the nth moments of the rod about its centre of mass. In evaluating the above moments we used the fact that

$$\overset{0}{h} = 1, \qquad \overset{1}{h} = 0, \qquad \int_{-k_1}^{k_2} \zeta^n h(\zeta)\, d\zeta = -\frac{1}{n+1} \overset{n+2}{h} \qquad (n = 0, 1, 2, \ldots). \tag{2.12}$$

The formulae (2.7) to (2.9) will be needed in the next section.

3. Solution of the linearized field equations

Using Galilean coordinates x_i we derive for the rotating rod an appropriate external solution of the linearized form of the field equations (1.1).

Let us consider, first of all, *any* isolated cohesive material distribution. For weak fields let us suppose that

$$g_{ik} = \eta_{ik} + \gamma_{ik} \tag{3.1}$$

where $\eta_{ik} = \eta^{ik} = \operatorname{diag}(-1, -1, -1, +1)$ and γ_{ik} are small. We introduce the auxiliary quantities γ^*_{ik} by

$$\gamma^*_{ik} = \gamma_{ik} - \tfrac{1}{2}\eta_{ik}\eta^{ab}\gamma_{ab} \tag{3.2}$$

so that

$$\gamma_{ik} = \gamma^*_{ik} - \tfrac{1}{2}\eta_{ik}\eta^{ab}\gamma^*_{ab} \tag{3.3}$$

and select (pseudo-) Galilean coordinates which satisfy the harmonic condition

$$\eta^{ab}\gamma^*_{ia,b} = 0 \tag{3.4}$$

where the comma means partial differentiation. The linearized form of the field equations

$$R_{ik} - \tfrac{1}{2}g_{ik}R = -8\pi T_{ik} \tag{3.5}$$

then reduces to the wave equations (Eddington 1924, § 57, Landau and Lifshitz 1962, § 101)

$$\eta^{ab}\gamma^*_{ik,ab} = -16\pi T_{ik} \tag{3.6}$$

their solution in Kirchhoff form for outgoing waves being

$$\gamma^*_{ik} = -4\int_V r^{*-1} T_{ik}(\tilde{x}_\alpha, t-r^*)\, d\tilde{x}_1\, d\tilde{x}_2\, d\tilde{x}_3, \qquad \eta^{ab}T_{ia,b} = 0. \tag{3.7}$$

Here the integration is taken over any fixed space volume V containing all the sources of the field and r^* is the distance of the point $\tilde{P}(\tilde{x}_\alpha)$, connected with the space element $d\tilde{x}_1\, d\tilde{x}_2\, d\tilde{x}_3$ of integration, from the field point $P(x_\alpha)$ under consideration. The second part of (3.7), immediately deducible from (3.4) and (3.6), simply expresses the law of conservation of energy and momentum in the linear approximation—a result already assumed in § 2 for the spinning rod in the calculation of the values (2.7) to (2.9) for the leading dimensionless moments $h_{ik/\sigma\rho\tau\ldots}$ of the corresponding T_{ik}.

It will be useful to have the formula in the first of (3.7) expressed as a multipole expansion in terms of r instead of r^* (where r is the *fixed* radius vector OP), before its application to the rotating rod. With the centre of mass taken as the origin O, the result is the (external) multipole wave solution, given below, of the linear approximation to (1.1) (Rotenberg 1964, appendix A), valid for r not less than the radius of the smallest sphere, with centre O, which can surround the entire physical distribution at all times; this expansion, in ascending powers of a, is written explicitly up to order a^3:

$$\overset{*}{\gamma}_{\alpha 4} = -4ma^2r^{-1}h_{\alpha\beta} - 4ma^3n_\sigma(r^{-1}h'_{\alpha\beta/\sigma} + r^{-2}h_{\alpha\beta/\sigma}) + mO(a^4) \tag{3.8}$$

$$\begin{aligned}\overset{*}{\gamma}_{\alpha 4} = {} & -4ma^2n_\sigma(r^{-1}h'_{\alpha 4/\sigma} + r^{-2}h_{\alpha 4/\sigma}) \\ & -2ma^3\{r^{-1}n_\sigma n_\rho h''_{\alpha 4/\sigma\rho} + (3n_\sigma n_\rho - \delta_{\sigma\rho})(r^{-2}h'_{\alpha 4/\sigma\rho} + r^{-3}h_{\alpha 4/\sigma\rho})\} + mO(a^4)\end{aligned} \tag{3.9}$$

$$\begin{aligned}\overset{*}{\gamma}_{44} = {} & -4mr^{-1}h_{44} - 2ma^2\{r^{-1}n_\sigma n_\rho h''_{44/\sigma\rho} + (3n_\sigma n_\rho - \delta_{\sigma\rho})(r^{-2}h'_{44/\sigma\rho} + r^{-3}h_{44/\sigma\rho})\} \\ & - \tfrac{2}{3}ma^3\{r^{-1}n_\sigma n_\rho n_\tau h'''_{44/\sigma\rho\tau} + 3r^{-2}n_\sigma(2n_\rho n_\tau - \delta_{\rho\tau})h''_{44/\sigma\rho\tau} \\ & + 3n_\sigma(5n_\rho n_\tau - 3\delta_{\rho\tau})(r^{-3}h'_{44/\sigma\rho\tau} + r^{-4}h_{44/\sigma\rho\tau})\} + mO(a^4)\end{aligned} \tag{3.10}$$

where

$$\eta^{ab}T_{ia,b} = 0. \tag{3.11}$$

In this expansion

$$n_\sigma = x_\sigma/r; \tag{3.12}$$

the dimensionless moments $h_{ik/\sigma\rho\tau\ldots}$ of T_{ik}, introduced from (2.6) and (2.4), are to be evaluated for the retarded time $u = t - r$ and a prime denotes differentiation with respect to u.

Inserting (2.7) to (2.9) in (3.8) to (3.10) we obtain for the rod the following multipole wave solution, valid for $r > \max\{ak_1, ak_2\}$, written explicitly up to order a^3:

$$\overset{*}{\gamma}_{ik} = m\overset{1*}{\gamma}_{ki} \tag{3.13}$$

where the non-zero $\overset{1*}{\gamma}_{ki}$ are

$$\begin{aligned}\overset{1*}{\gamma}_{11} = {} & 4a^2\overset{2}{h}\omega^2r^{-1}(1-2s^2) \\ & -2a^3\overset{3}{h}[\omega^3r^{-1}\{\lambda(7s-9s^3) + \mu(8c-9c^3)\} + \omega^2r^{-2}\{\lambda(2c-3c^3) - \mu(s-3s^3)\}] + O(a^4)\end{aligned} \tag{3.14}$$

$$\overset{1*}{\gamma}_{12} = 8a^2\overset{2}{h}\omega^2r^{-1}sc + 6a^3\overset{3}{h}[\omega^3r^{-1}\{-\lambda(2c-3c^3) + \mu(2s-3s^3)\} + \omega^2r^{-2}(\lambda sc^2 + \mu s^2c)] + O(a^4) \tag{3.15}$$

$$\begin{aligned}\overset{1*}{\gamma}_{22} = {} & -4a^2\overset{2}{h}\omega^2r^{-1}(1-2s^2) \\ & +2a^3\overset{3}{h}[\omega^3r^{-1}\{\lambda(8s-9s^3) + \mu(7c-9c^3)\} + \omega^2r^{-2}\{\lambda(c-3c^3) - \mu(2s-3s^3)\}] + O(a^4)\end{aligned} \tag{3.16}$$

$$\begin{aligned}\overset{1}{\gamma}{}^*_{14} = {} & -4a^2\overset{2}{h}[\omega^2 r^{-1}\{\lambda(1-2s^2)+2\mu sc\}+\omega r^{-2}(\lambda sc+\mu s^2)] \\ & +2a^3\overset{3}{h}[\omega^3 r^{-1}\{\lambda^2(7s-9s^3)+2\lambda\mu(7c-9c^3)-3\mu^2(2s-3s^3)\} \\ & +\omega^2 r^{-2}\{(3\lambda^2-1)(2c-3c^3)-6\lambda\mu(2s-3s^3)-3(3\mu^2-1)s^2c\} \\ & -\omega r^{-3}\{(3\lambda^2-1)sc^2+6\lambda\mu s^2c+(3\mu^2-1)s^3\}]+\mathrm{O}(a^4)\end{aligned} \tag{3.17}$$

$$\begin{aligned}\overset{1}{\gamma}{}^*_{24} = {} & 4a^2\overset{2}{h}[\omega^2 r^{-1}\{-2\lambda sc+\mu(1-2s^2)\}+\omega r^{-2}(\lambda c^2+\mu sc)] \\ & +2a^3\overset{3}{h}[\omega^3 r^{-1}\{3\lambda^2(2c-3c^3)-2\lambda\mu(7s-9s^3)-\mu^2(7c-9c^3)\} \\ & +\omega^2 r^{-2}\{-3(3\lambda^2-1)sc^2-6\lambda\mu(2c-3c^3)+(3\mu^2-1)(2s-3s^3)\} \\ & +\omega r^{-3}\{(3\lambda^2-1)c^3+6\lambda\mu sc^2+(3\mu^2-1)s^2c\}]+\mathrm{O}(a^4)\end{aligned} \tag{3.18}$$

$$\begin{aligned}\overset{1}{\gamma}{}^*_{44} = {} & -4r^{-1}+2a^2\overset{2}{h}[2\omega^2 r^{-1}\{(\lambda^2-\mu^2)(1-2s^2)+4\lambda\mu sc\}+6\omega r^{-2}\{(\lambda^2-\mu^2)sc-\lambda\mu(1-2s^2)\} \\ & -r^{-3}\{(3\lambda^2-1)c^2+6\lambda\mu sc+(3\mu^2-1)s^2\}] \\ & -2a^3\overset{3}{h}[\omega^3 r^{-1}\{\lambda^3(7s-9s^3)+\lambda^2\mu(20c-27c^3)-\lambda\mu^2(20s-27s^3)-\mu^3(7c-9c^3)\} \\ & +\omega^2 r^{-2}\{3\lambda(2\lambda^2-1)(2c-3c^3)-\mu(6\lambda^2-1)(7s-9s^3)-\lambda(6\mu^2-1)(7c-9c^3) \\ & +3\mu(2\mu^2-1)(2s-3s^3)\} \\ & +3\omega r^{-3}\{-\lambda(5\lambda^2-3)sc^2-\mu(5\lambda^2-1)(2c-3c^3)+\lambda(5\mu^2-1)(2s-3s^3)+\mu(5\mu^2-3)s^2c\} \\ & +r^{-4}\{\lambda(5\lambda^2-3)c^3+3\mu(5\lambda^2-1)sc^2+3\lambda(5\mu^2-1)s^2c+\mu(5\mu^2-3)s^3\}]+\mathrm{O}(a^4).\end{aligned} \tag{3.19}$$

In the above formulae

$$(\lambda, \mu, \nu) = n_\alpha = x_\alpha/r \tag{3.20}$$

and instead of (2.10)

$$s = \sin \omega u, \qquad c = \cos \omega u, \qquad u = t-r. \tag{3.21}$$

It can be verified by lengthy, but straightforward, calculations that the solution (3.13) to (3.19) actually satisfies the linearized field equations (3.4) and

$$\eta^{ab}\gamma^*_{ik,ab} = 0 \tag{3.22}$$

for free space.

We notice that the values for γ^*_{ik} given by the multipole wave solution (3.8) to (3.10) of the linearized field equations, for any isolated coherent material source of mass m, are linear in m. This suggests that an appropriate exact external solution of (1.1) for any such source is expansible as an infinite power series in m:

$$\gamma^*_{ik} = \sum_{p=1}^{\infty} m^p \overset{p}{\gamma}{}^*_{ki} \tag{3.23}$$

where $\overset{p}{\gamma}{}^*_{ik}$ are independent of m, with the set of coefficients $\overset{1}{\gamma}{}^*_{ki}$ of m given by that in (3.8) to (3.10).† Inserting (3.23) in (1.1) and equating to zero the coefficients of m^p in the result yields the *pth approximation*

$$\overset{p}{R}_{ik} = 0. \tag{3.24}$$

It consists of ten second-order differential equations of the forms

$$\Phi_{lm}(\overset{p}{\gamma}{}^*_{ik}) = \overset{p}{\Psi}_{lm}(\overset{q}{\gamma}{}^*_{ik}) \quad (q \leqslant p-1) \tag{3.25}$$

† Since, by virtue of (3.8) to (3.10), the set $\overset{1}{\gamma}{}^*_{ki}$ itself is equal to an infinite expansion in ascending powers of a, a double infinite series in ascending powers of m and a for the metric has been considered by Bonnor in the study of gravitational waves (see Bonnor 1959, 1963, Bonnor and Rotenberg 1966).

where the left-hand sides are linear in $\overset{p}{\gamma}{}^{*}_{ik}$ (and their derivatives), and the right-hand sides are non-linear in $\overset{q}{\gamma}{}^{*}_{ik}$ (and their derivatives) known from solutions of the previous, qth, approximations.

The appropriate exact external solution for the spinning rod, although not explicitly known beyond the first approximation, will be referred to in the next section as (3.23) and (3.14) to (3.19). Furthermore, it will be assumed that the metric for the rod can be expanded in powers of m as in (3.23) throughout the entire space–time, *including the neighbourhood of the rod*, and that the *form* of the appropriate solution of the linear approximation is as in (3.13) throughout all space–time.

4. Calculation of the rates of energy and momentum flow

In obtaining expressions for the rates of flow of energy and momentum from the rotating rod we shall employ both (i) the ordinary energy–momentum tensor with use of what is known as 'Synge's argument' and (ii) the energy pseudo-tensor, and then compare results.

4.1. *Use of the energy–momentum tensor*

Here we use an argument due originally to Synge (1960, chap. IV, § 6), which is lucidly explained by Bonnor (1959, § 11). It concerns approximate solutions of the field equations for any gravitating source; for an isolated coherent source Σ it goes as follows.

Let us take any approximate solution of (1.1) corresponding to the source Σ. Substituting the solution into

$$G_k{}^i \overset{\text{def}}{=} R_k{}^i - \tfrac{1}{2}\delta_k{}^i R = -8\pi T_k{}^i \tag{4.1}$$

we can obtain an expression for the energy tensor $T_k{}^i$ which corresponds to it. This will not vanish anywhere, unlike the case of an appropriate exact solution in which $T_k{}^i$ vanishes everywhere except for the region occupied by Σ. Thus for the approximate solution under consideration a continuous distribution of matter (and stresses) will appear throughout space–time, and this distribution† together with Σ may be regarded as representing the 'source' of the approximate solution; in other words, the approximate solution of (1.1) may be considered as an exact solution of (4.1) corresponding to the augmented source.

Now, let us apply this argument to the approximate wave solution (3.13) to (3.19) for the spinning rod introduced in § 2. After some lengthy calculation we obtain‡

$$-8\pi T_k{}^i = m^2\left\{r^{-2}\lambda^i\lambda_k \sum_{s=4}^{\infty} a^s K_{(s)} + L_k{}^i + \mathrm{O}(r^{-3})\right\} + \mathrm{O}(m^3). \tag{4.2}$$

In this

$$\lambda_i = u_{,i} = (-n_\alpha, 1), \qquad \lambda^i = (n_\alpha, 1) \tag{4.3}$$

so that

$$\lambda_i = \eta_{ia}\lambda^a, \qquad \lambda^i = \eta^{ia}\lambda_a, \qquad \lambda^a\lambda_a = 0; \tag{4.4}$$

$$\left.\begin{aligned} K_{(4)} &= 16\omega^6 \overset{2}{h}{}^2\{2-2(\bar{\lambda}^2+\bar{\mu}^2)+(\bar{\lambda}^4-4\bar{\lambda}^2\bar{\mu}^2+\bar{\mu}^4)\} \\ K_{(5)} &= 2\omega^7 \overset{2}{h}\overset{3}{h}\bar{\mu}\{54-(45\bar{\lambda}^2+59\bar{\mu}^2)+(117\bar{\lambda}^4-197\bar{\lambda}^2\bar{\mu}^2+28\bar{\mu}^4)\}, \ldots \end{aligned}\right\} \tag{4.5}$$

† Hereafter to be referred to as the 'phoney' matter corresponding to the approximate solution.

‡ In terms of $\overset{1}{\gamma}{}^{*}_{ik}$ that constitutes the multipole wave solution (3.13) and (3.8) to (3.10) of the linear approximation for any isolated coherent material source, the value of the Einstein tensor $G_k{}^i$ is (Rotenberg, to be published).

$$\begin{aligned} G_k{}^i = m^2[\lambda^i\lambda_k(&\tfrac{1}{4}\overset{1}{\gamma}{}^{*}\overset{1}{\gamma}{}^{*}_{,44} + \tfrac{1}{8}\overset{1}{\gamma}{}^{*}_{,4}\overset{1}{\gamma}{}^{*}_{,4} - \tfrac{1}{2}\overset{1}{\gamma}{}^{*ab}\overset{1}{\gamma}{}^{*}_{ab,44} - \tfrac{1}{4}\overset{1}{\gamma}{}^{*ab}{}_{,4}\overset{1}{\gamma}{}^{*}_{ab,4}) \\ &+ r^{-1}\eta^{ia}\{2(\overset{1}{\gamma}{}^{*}_{ak} - \lambda_a\overset{1}{\gamma}{}^{*}_{k4} - \lambda_k\overset{1}{\gamma}{}^{*}_{a4}) + (\eta_{ak} - \lambda_a\delta_{k4} - \lambda_k\delta_{a4})\overset{1}{\gamma}{}^{*}\}_{,44} + \mathrm{O}(r^{-3})] + \mathrm{O}(m^3) \end{aligned}$$

where

$$\overset{1}{\gamma}{}^{*} = -\eta^{ab}\overset{1}{\gamma}{}^{*}_{ab}, \qquad \overset{1}{\gamma}{}^{*ik} = \eta^{ia}\eta^{kb}\overset{1}{\gamma}{}^{*}_{ab}.$$

where

$$\bar{\lambda} = \lambda \cos \omega u + \mu \sin \omega u, \qquad \bar{\mu} = -\lambda \sin \omega u + \mu \cos \omega u; \tag{4.6}$$

and

$$L_k{}^i = r^{-1}\eta^{ia}\{2(\overset{1}{\gamma}{}^*_{ak} - \lambda_a \overset{1}{\gamma}{}^*_{k4} - \lambda_k \overset{1}{\gamma}{}^*_{a4} + \lambda_a\lambda_k \overset{1}{\gamma}{}^*_{44}) + (\eta_{ak} - \lambda_a\delta_{k4} - \lambda_k\delta_{a4} + \lambda_a\lambda_k)\overset{1}{\gamma}{}^*\}_{,44} \tag{4.7}$$

where

$$\overset{1}{\gamma}{}^* = -\eta^{ab}\overset{1}{\gamma}{}^*_{ab}. \tag{4.8}$$

Thus, for $r > \max\{ak_1, ak_2\}$, the approximate solution (3.13) represents exactly the distribution which is composed of the rod and the tenuous ('phoney') matter described by the energy tensor $T_k{}^i$ given by (4.2).

From (4.2) the rates at which mass and momentum of the 'phoney' matter flow out of a large sphere S, with centre the origin O and radius r, can be readily calculated as below.

It can easily be shown that $L_k{}^i$ in (4.7) satisfies (Rotenberg, to be published)

$$L_\beta{}^\alpha n_\alpha = \mathrm{O}(r^{-3}), \qquad L_4{}^i = 0. \tag{4.9}$$

Hence, from (4.2), we have

$$\frac{dJ_k{}^{(\mathrm{P})}}{dt} \equiv \int_S \mathfrak{T}_k{}^\alpha n_\alpha \, dS = -\frac{1}{8\pi} m^2 \sum_{s=4}^{\infty} a^s \int_\Omega \lambda_k K_{(s)} \, d\Omega + \mathrm{O}(r^{-1}) + \mathrm{O}(m^3) \tag{4.10}$$

where $J_k{}^{(\mathrm{P})}(t)$ is the covariant 4-momentum of the 'phoney' matter that has flowed out of the sphere S by time t, and where the integration on the extreme right is to be carried out over the surface Ω of a unit sphere with centre O. Using (4.5), (4.6), (4.3) and (3.20) in (4.10), we obtain, for the rate at which the covariant 4-momentum of 'phoney' matter flows out of S,†

$$\left.\begin{aligned} \frac{dJ_\alpha{}^{(\mathrm{P})}}{dt} &= m^2\left\{\frac{172}{35} a^5\omega^7 \overset{2}{h}\overset{3}{h}(-\sin \omega u, \cos \omega u, 0) + \mathrm{O}(a^6)\right\} + \mathrm{O}(m^3) \\ \frac{dJ_4{}^{(\mathrm{P})}}{dt} &= m^2\left\{-\frac{32}{5} a^4\omega^6 \overset{2}{h}{}^2 + \mathrm{O}(a^6)\right\} + \mathrm{O}(m^3) \end{aligned}\right\} \tag{4.11}$$

ignoring terms of order r^{-1} for large r.

The result (4.11) implies that, to maintain the field (3.13), i.e. to maintain the constancy of the 4-momentum of the rod, matter must be extracted out of S with 4-momentum flowing out at the rate (4.11). But in the actual physical situation no such matter exists. It therefore follows that the 4-momentum of the rod must increase (or, to be more precise, vary) at the rate given by (4.11) with replacement of u by t. Hence at any time t the 4-momentum J_k of the rod varies at the rate‡

$$\left.\begin{aligned} \frac{dJ_\alpha}{dt} &= m^2\left\{\frac{172}{35} a^5\omega^7 \overset{2}{h}\overset{3}{h}(-\sin \omega t, \cos \omega t, 0) + \mathrm{O}(a^6)\right\} + \mathrm{O}(m^3) \\ \frac{dJ_4}{dt} &= m^2\left\{-\frac{32}{5} a^4\omega^6 \overset{2}{h}{}^2 + \mathrm{O}(a^6)\right\} + \mathrm{O}(m^3) \end{aligned}\right\} \tag{4.12}$$

on account of the outward flow of 4-momentum of the wave field.

If we were to pursue the approximation to the metric for the rod up to the second order of m, i.e. solve (3.24) ($p = 2$) for $\overset{2}{\gamma}{}^*_{ik}$ in terms of the known $\overset{1}{\gamma}{}^*_{ik}$, we would most probably find expressions in $\overset{2}{\gamma}{}^*_{ik}$ representing a rate of change of 4-momentum of the rod equal and

† The notation $\mathrm{O}(m^p)$ in (4.11) and elsewhere denotes any expression in the form of a power series, in both m and a, comprising terms of orders $m^q a^s$ ($q \geqslant p$, $s \geqslant 0$).

‡ dJ_3/dt most probably vanishes on physical grounds.

opposite to the rate of 4-momentum flux of the wave field, given by the leading (m^2) contributions in (4.12). (We would then obviously find that no flux of 4-momentum of 'phoney' matter calculated from the second approximation existed in that approximation.) A confirmation of this is intended by the author for a future paper.

For the contravariant 4-momentum J^k we have, from (4.12),

$$\left.\begin{aligned} \frac{dJ^\alpha}{dt} &= m^2\left\{\frac{172}{35}a^5\omega^7\overset{2}{h}\overset{3}{h}(\sin\omega t,\ -\cos\omega t,\ 0)+\mathrm{O}(a^6)\right\}+\mathrm{O}(m^3) \\ \frac{dJ^4}{dt} &= m^2\left\{-\frac{32}{5}a^4\omega^6\overset{2}{h}{}^2+\mathrm{O}(a^6)\right\}+\mathrm{O}(m^3) \end{aligned}\right\} \tag{4.13}$$

since, from (3.1), (3.3) and (3.23),

$$J^\alpha = -J_\alpha+\mathrm{O}(m^3), \qquad J^4 = J_4+\mathrm{O}(m^3). \tag{4.14}$$

Neglecting terms of order a^6 and m^3 we obtain the following two results:

(i) The spinning rod loses mass at the steady rate

$$-\frac{dm}{dt} = \frac{32}{5}\overset{2}{I}{}^2\omega^6 \tag{4.15}$$

where $\overset{n}{I}$ is its nth moment about its centre of mass, as in (2.11). This is the result obtained by Eddington (1924, p. 248) using a different method, and by Weber (1961, § 7.6) using the pseudo-tensor employed later on in this section.

(ii) The linear momentum of the rod varies cyclically at the rate

$$\mathbf{F} \overset{\mathrm{def}}{=} \frac{d\mathbf{J}}{dt} = \frac{172}{35}\overset{2}{I}\overset{3}{I}\omega^7(\sin\omega t,\ -\cos\omega t,\ 0). \tag{4.16}$$

This is equivalent to a centripetal force **F** acting on the rod perpendicularly to it through its centre of mass. The effect of the gravitational waves is to supply a centrifugal force $-\mathbf{F}$ perpendicular to the rod through its centre of mass. Thus the centre of rotation of the rod is not its centre of mass, but a fixed point O distant

$$\delta = \frac{172}{35}\frac{\overset{2}{I}\overset{3}{I}\omega^5}{m} \tag{4.17}$$

from the axis of symmetry of the rod on the line perpendicular to the rod through its centre of mass. In other words, the rod moves as a tangent of the circle, centre O and radius δ, rotating around the circle with angular velocity ω; the centre of mass of the rod is always kept as the point of contact between the rod and the circle.

Obtaining result (ii) was the main object of the paper.

4.2. *Use of the pseudo-tensor*

The energy tensor density $\mathfrak{T}_k{}^i$ and the pseudo-tensor $\mathfrak{t}_k{}^i$ are connected by the conservation law (Eddington 1924, § 59, Tolman 1934, § 87)

$$\frac{\partial}{\partial x_i}(\mathfrak{T}_k{}^i+\mathfrak{t}_k{}^i) = 0 \tag{4.18}$$

where x_i are the Galilean coordinates used throughout. If S is a two-dimensional sphere, centre O and large radius r, containing space volume V and enclosing all the sources of the field, then from (4.18)

$$-\frac{d}{dt}\int_V(\mathfrak{T}_k{}^4+\mathfrak{t}_k{}^4)\,dv = \int_S(\mathfrak{T}_k{}^\alpha+\mathfrak{t}_k{}^\alpha)n_\alpha\,dS \tag{4.19}$$

where $n_\alpha = x_\alpha / r$ are the components of the outward space normal to S. Since $\mathfrak{T}_k{}^i = 0$ on S, this gives

$$-\frac{d}{dt}\int_V (\mathfrak{T}_k{}^4 + \mathfrak{t}_k{}^4)\,dv = \int_S \mathfrak{t}_k{}^\alpha n_\alpha\,dS \tag{4.20}$$

which is the usual integral theorem referring to the energy and momentum of the field; the surface integral term on the right represents the rate of flux of 4-momentum J_k of the field out of S. Thus we may write

$$\frac{dJ_k}{dt} = \int_S \mathfrak{t}_k{}^\alpha n_\alpha\,dS \tag{4.21}$$

and we now calculate the surface integral on the right for the rotating rod.

For the pseudo-tensor we use the usual formula (Eddington 1924, § 59, Tolman 1934, § 87)

$$16\pi \mathfrak{t}_k{}^i = \mathfrak{g}^{ab}{}_{,k}\Gamma_{ab}{}^i - \mathfrak{g}^{ai}{}_{,k}\Gamma_{ab}{}^b + \delta_k{}^i \mathfrak{g}^{lm}(\Gamma_{lb}{}^a\Gamma_{ma}{}^b - \Gamma_{lm}{}^a\Gamma_{ab}{}^b) \tag{4.22}$$

and substitute into it the external wave solution relevant to the rod, represented by (3.23) and (3.14) to (3.19). We evaluate only the leading term in the resulting expansion in ascending powers of m for $\mathfrak{t}_k{}^i$, as we did in calculation of $T_k{}^i$; it is of order m^2, as the formula (4.22) consists essentially of products of the $\gamma^*_{ik,l}$, which by virtue of (3.23) are of order m. It makes no difference if we substitute into (4.22) the exact solution (3.23) or the approximate solution (3.13) as far as this leading term is concerned, since a choice between the two solutions only affects contributions of order m^3 and higher in $\mathfrak{t}_k{}^i$. Hence the reason for not proceeding further than the linear approximation to the wave solution in § 3.

The result of substitution of (3.13) into (4.22) is, after some calculation,†

$$\mathfrak{t}_k{}^i = m^2\left\{r^{-2}\lambda^i\lambda_k \sum_{s=4}^{\infty} a^s w_{(s)} + \mathrm{O}(r^{-3})\right\} + \mathrm{O}(m^3) \tag{4.23}$$

where λ_i, λ^i are as in (4.3) and where

$$\left.\begin{aligned} w_{(4)} &= \frac{4}{\pi}\,\omega^6 \overset{2}{h}{}^2\{1-(\bar\lambda^2+\bar\mu^2)+\bar\lambda^2\bar\mu^2\} \\ w_{(5)} &= \frac{1}{\pi}\,\omega^7 \overset{2}{h}\overset{3}{h}\bar\mu\{12-(13\bar\lambda^2+12\bar\mu^2)-(7\bar\lambda^4-20\bar\lambda^2\bar\mu^2)\}, \qquad \ldots \end{aligned}\right\} \tag{4.24}$$

with

$$\bar\lambda = \lambda\cos\omega u + \mu\sin\omega u, \qquad \bar\mu = -\lambda\sin\omega u + \mu\cos\omega u. \tag{4.25}$$

Using (4.23) and (4.3), (4.21) gives

$$\frac{dJ_k}{dt} = m^2 \sum_{s=4}^{\infty} a^s \int_\Omega \lambda_k w_{(s)}\,d\Omega + \mathrm{O}(r^{-1}) + \mathrm{O}(m^3) \tag{4.26}$$

for the rate of flow of J_k out of the sphere S, where Ω is the unit sphere with centre O. Then

† In terms of $\overset{1}{\gamma}{}^*_{ik}$ constituting the multipole wave solution (3.13) and (3.8) to (3.10) for any isolated cohesive mechanical system, the value of $\mathfrak{t}_k{}^i$ is (Rotenberg, to be published)

$$16\pi \mathfrak{t}_k{}^i = m^2\{\lambda^i\lambda_k(\tfrac{1}{2}\overset{1}{\gamma}{}^{*ab}{}_{,4}\overset{1}{\gamma}{}^*_{ab,4} - \tfrac{1}{4}\overset{1}{\gamma}{}^*_{,4}\overset{1}{\gamma}{}^*_{,4}) + \mathrm{O}(r^{-3})\} + \mathrm{O}(m^3)$$

where

$$\overset{1}{\gamma}{}^* = -\eta^{ab}\overset{1}{\gamma}{}^*_{ab}, \qquad \overset{1}{\gamma}{}^{*ik} = \eta^{ia}\eta^{kb}\overset{1}{\gamma}{}^*_{ab}.$$

from (4.24), (4.25), (4.3) and (3.20) we have, ignoring terms of order r^{-1} for large r,

$$\left.\begin{aligned} \frac{dJ_\alpha}{dt} &= m^2\left\{\frac{464}{105}\,a^5\omega^7\overset{2}{h}\overset{3}{h}(\sin\omega u,\ -\cos\omega u,\ 0)+\mathrm{O}(a^6)\right\}+\mathrm{O}(m^3) \\ \frac{dJ_4}{dt} &= m^2\left\{\frac{32}{5}\,a^4\omega^6\overset{2}{h}{}^2+\mathrm{O}(a^6)\right\}+\mathrm{O}(m^3) \end{aligned}\right\}. \tag{4.27}$$

Ignoring contributions of order a^6 and m^3, we obtain from (4.27) and (4.14) the following two results, similar to (4.15) and (4.16), respectively, but the second differing numerically from (4.16):

(i) On account of J^4 flux, the rod loses mass at the constant rate

$$-\frac{dm}{dt} = \frac{32}{5}\overset{2}{I}{}^2\omega^6. \tag{4.28}$$

(ii) On account of J^β flux, the linear momentum of the rod varies cyclically at the rate

$$\mathbf{F} \overset{\text{def}}{=} \frac{d\mathbf{J}}{dt} = \frac{464}{105}\overset{2}{I}\overset{3}{I}\omega^7(\sin\omega t,\ -\cos\omega t,\ 0). \tag{4.29}$$

This corresponds to a centripetal force $\mathbf{F}$ acting on the rod perpendicularly to it through its centre of mass. Thus the centre of rotation of the rod does not coincide with its centre of mass, but is a fixed point O distant

$$\delta = \frac{464}{105}\,\frac{\overset{2}{I}\overset{3}{I}\omega^5}{m} \tag{4.30}$$

from the axis of symmetry of the rod on the line perpendicular to the rod through its centre of mass.

The reason for the discrepancy between the two numerical results (4.16) and (4.29) is the following.

Let us write (4.20) as

$$\frac{d}{dt}\int_V \mathfrak{T}_k{}^4\,dv+\frac{d}{dt}\int_V \mathfrak{t}_k{}^4\,dv = -\frac{dJ_k}{dt} \tag{4.31}$$

with J_k as in (4.21). Then the rate at which the 4-momentum of the rod varies, given by the first expression on the left of (4.31), is equal and opposite to dJ_k/dt only if†

$$\frac{d}{dt}\int_V \mathfrak{t}_k{}^4\,dv = 0. \tag{4.32}$$

That this is true (up to the term in m^2a^5) *only if* $k = 3, 4$ will now be shown, under the assumption that the metric for the rod can be expanded in the form (3.23) throughout space–time, including the neighbourhood of the origin.

Let the approximate metric (3.13) represent *exactly* the material distribution made up of the rotating rod, of constant mass and momentum, and the matter that has to be infused from outside into the neighbourhood of the rod in such a way as to secure the constancy of its mass and momentum. Then this metric must satisfy

$$\frac{d}{dt}\int_V \mathfrak{T}_k{}^4\,dv = 0 \tag{4.33}$$

† This statement should be compared with the criticism Peres (1960) makes on the proof by Infeld (1959) that gravitational radiation does not exist.

exactly. Thus, from (4.19) we have for this metric

$$\frac{d}{dt}\int_V \mathfrak{t}_k{}^4\, dv = -\int_S (\mathfrak{T}_k{}^\alpha + \mathfrak{t}_k{}^\alpha) n_\alpha\, dS = \frac{52}{105} \overset{23}{I\!I} \omega^7 U_k + m^2 O(a^6) + O(m^3) \tag{4.34}$$

where

$$U_k = \sin \omega u, \ -\cos \omega u, 0, 0 \tag{4.35}$$

by virtue of (4.10), (4.11), (4.21) and (4.27). Formula (4.34) is true not only for the approximate metric (3.13) but also for the exact metric (3.23). For, as mentioned earlier, the difference between the values of $\mathfrak{t}_k{}^i$ for these two metrics, and therefore the difference between the corresponding values of the extreme left of (4.34), is of order m^3. Hence (4.34) and (4.35) show that (4.32) is true up to the term in $m^2 a^5$ *only if* $k = 3, 4$.

To obtain the value of the left-hand side of (4.32) directly, without the help of Synge's argument, necessitates finding a suitable solution of the linear approximation for the whole of space–time, including the region $r \leqslant \max\{ak_1, ak_2\}$. This involves matching an internal solution with an external solution for a moving system, not spherically symmetric, and therefore extremely difficult to achieve. Such is an example of the inadequacy of the pseudo-tensor for use in calculation of the flux of 4-momentum from isolated radiating material systems.

Thus, as far as variation of 3-momentum of the rod is concerned, the result (4.16) should be taken as the correct one in preference to the result (4.29). In any case, the former result was achieved by the use of a real tensor having covariant properties.

Acknowledgments

The author offers his deep gratitude to Professor W. B. Bonnor, whose helpful criticisms led to certain improvements in presentation of the text.

References

Bergmann, P. G., 1942, *Introduction to the Theory of Relativity* (Englewood Cliffs, N.J.: Prentice-Hall).
Bonnor, W. B., 1959, *Phil. Trans. Roy. Soc.* A, **251**, 233–71.
—— 1963, *Brit. J. Appl. Phys.*, **14**, 555–62.
Bonnor, W. B., and Rotenberg, M. A., 1966, *Proc. Roy. Soc.* A, **289**, 247–74.
Clark, G. L., 1947, *Proc. Camb. Phil. Soc.*, **43**, 164–77.
Eddington, A. S., 1924, *The Mathematical Theory of Relativity* (London: Cambridge University Press).
Einstein, A., 1916, *S.B. preuss. Akad. Wiss.*, 688.
—— 1918, *S.B. preuss. Akad. Wiss.*, 154.
Infeld, L., 1959, *Ann. Phys., N.Y.*, **6**, 341–67.
Landau, L. D., and Lifshitz, E. M., 1962, *The Classical Theory of Fields* (Oxford: Pergamon Press).
Peres, A., 1960, *Nuovo Cim.*, **15**, 351–69.
Rotenberg, M. A., 1964, *Ph.D. Thesis*, University of London.
Synge, J. L., 1960, *Relativity: the General Theory* (Amsterdam: North-Holland).
Tolman, R. C., 1934, *Relativity, Thermodynamics and Cosmology* (Oxford: Clarendon Press).
Weber, J., 1961, *General Relativity and Gravitational Waves* (New York: Interscience).